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ORIGINAL PAPER

Optimal induced norm computation of discrete H_∞ control systems with time-delays

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Abstract Time-delays in state or control can never be eliminated in many discrete systems, like computer controlled systems. Introducing an extended state vector, the original equations with time-delays can be transformed into standard equations without time-delays. Then the theory and methods of usual discrete system can be applied. Based on analogies between structural mechanics and optimal control theory, the optimal norm corresponding to the fundamental frequency of structural vibration, which is a Rayleigh-quotient problem, can be solved by extended Wittrick–Williams algorithm. Numerical results disclose that the optimal norm does not increase monotonously with time-delays and can be decreased effectively by selecting appropriate time delays.

Keywords Discrete systems with time-delays · Optimal H_∞ induced norm · Extended W–W algorithm

Introduction

Time delay exists commonly in dynamic systems due to measurement, transmission and transport lags, computational delays or unmodelled inertias of system components. Time delay has been generally regarded as a main source of instability and poor performance. Therefore, the research on the problem of H_∞ control with time-delays is very important for both theory and practice. Optimal discrete H_∞ control with time-delays is investigated in this paper. Introducing extended state vectors [1], the original equations with time-delays are transformed into standard equations without time-delays. Since the controller is obtained directly from the time-delay equations, system stability can be guaranteed easily. Based on the analogy between structural mechanics and optimal control theory, this paper adopts the extended

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Wittrick–Williams (W–W) algorithm [2,3] to compute optimal induced norm of discrete H_∞ control systems with time-delays. Numerical results demonstrate the influence of time-delays on the optimal norm.

1 Problem formulation

Considering the following uncertain discrete system with state time-delays:

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_0\boldsymbol{x}_k + \boldsymbol{A}_1\boldsymbol{x}_{k-1} + \cdots + \boldsymbol{A}_m\boldsymbol{x}_{k-m} + \boldsymbol{B}_0\boldsymbol{u}_k + \boldsymbol{D}_0\boldsymbol{w}_k, \quad \boldsymbol{x}_k = \boldsymbol{0}, \quad \text{when } k = -m, \ldots, 0, \tag{1.1}$$

$$\boldsymbol{z}_k = \boldsymbol{H}_0\boldsymbol{x}_k + \boldsymbol{N}_0\boldsymbol{u}_k, \tag{1.2}$$

where $k \in [0, N-1]$, $\mathbf{x}_k \in R^n$, $\boldsymbol{w}_k \in R^l$, $\boldsymbol{u}_k \in R^m$ and $\boldsymbol{z}_k \in R^p$ are the state vector, disturbance input, control vector and output control, respectively, and $m \geq 0$ is a known constant delay. $\boldsymbol{A}_0$, $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$, $\boldsymbol{B}_0$, $\boldsymbol{D}_0$, $\boldsymbol{H}_0$ and N_0 are system matrices with appropriate dimensions. It is also assumed that $\boldsymbol{N}_0^{\mathrm{T}}[\boldsymbol{H}_0\ \boldsymbol{N}_0] = [\boldsymbol{0}\ \boldsymbol{I}]$, $\boldsymbol{Q}_0 = \boldsymbol{H}_0^{\mathrm{T}}\boldsymbol{H}_0$.

Referring to the typical H_∞ control problem, the object of H_∞ control with time-delays is to find an optimal control strategy $\boldsymbol{u}_k^* = \Gamma(\boldsymbol{x}_k, \ldots, \boldsymbol{x}_{k-m})$ in the square summable space $\mathcal{L}_2[0, N-1]$ such that

$$\frac{1}{2}\sum_{k=0}^{N-1} \boldsymbol{z}_k^{\mathrm{T}}\boldsymbol{z}_k + \frac{1}{2}\boldsymbol{x}_N^{\mathrm{T}}\boldsymbol{S}_N\boldsymbol{x}_N < \frac{1}{2}\gamma^2\sum_{k=0}^{N-1} \boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{w}_k, \quad \gamma_{\mathrm{opt}} = \max_{\boldsymbol{w}}\min_{\boldsymbol{u}} \gamma^2, \tag{1.3}$$

where $\boldsymbol{w} \in \mathcal{L}_2[0, N-1]$, $\boldsymbol{S}_N$ is a symmetric semi-positive-definite matrix, γ_{opt} is the optimal induced norm of discrete H_∞ control system with time-delays, which ensures the existence of the controller of the system. So the solution of γ_{opt} is very important for both design and analysis.

2 Standardization of system equation

Introducing the extended vector $\bar{\boldsymbol{x}}_k = \{\boldsymbol{x}_k^{\mathrm{T}}, \boldsymbol{x}_{k-1}^{\mathrm{T}}, \ldots, \boldsymbol{x}_{k-m}^{\mathrm{T}}\}^{\mathrm{T}}$ and the extended matrices

$$\overline{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_m \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{bmatrix}, \qquad \overline{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}_0 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \qquad \overline{\boldsymbol{D}} = \begin{bmatrix} \boldsymbol{D}_0 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$
$$\overline{\boldsymbol{H}} = \begin{bmatrix} \boldsymbol{H}_0 & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \quad \overline{\boldsymbol{N}} = \boldsymbol{N}_0. \tag{2.1}$$

Then the system equations (1.1) and (1.2) are standardized as follows

$$\bar{\boldsymbol{x}}_{k+1} = \overline{\boldsymbol{A}}\bar{\boldsymbol{x}}_k + \overline{\boldsymbol{B}}\boldsymbol{u}_k + \overline{\boldsymbol{D}}\boldsymbol{w}_k, \quad \bar{\boldsymbol{x}}_0 = \{\mathbf{0}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}, \cdots, \mathbf{0}^{\mathrm{T}}\}^{\mathrm{T}} \tag{2.2}$$

$$\boldsymbol{z}_k = \overline{\boldsymbol{H}}\bar{\boldsymbol{x}}_k + \overline{\boldsymbol{N}}\boldsymbol{u}_k. \tag{2.3}$$

Obviously, the features of $\overline{\boldsymbol{N}}^{\mathrm{T}}[\overline{\boldsymbol{H}}\ \overline{\boldsymbol{N}}] = [\mathbf{0}\ \mathbf{I}]$ and $\overline{\boldsymbol{Q}} = \overline{\boldsymbol{H}}^{\mathrm{T}}\overline{\boldsymbol{H}}$ are still maintained, where $\overline{\boldsymbol{Q}} = \mathrm{diag}([\boldsymbol{Q}_0, \mathbf{0}, \ldots, \mathbf{0}])$.

The above description about optimal H_∞ control with time-delays is also standardized to find an optimal control strategy $\boldsymbol{u}_k^* = \Gamma(\bar{\boldsymbol{x}}_k)$ such that

$$\frac{1}{2}\sum_{k=0}^{N-1} \boldsymbol{z}_k^{\mathrm{T}}\boldsymbol{z}_k + \frac{1}{2}\bar{\boldsymbol{x}}_N^{\mathrm{T}}\overline{\boldsymbol{S}}_N\bar{\boldsymbol{x}}_N < \frac{1}{2}\gamma^2\sum_{k=0}^{N-1}\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{w}_k, \quad \gamma_{\mathrm{opt}} = \max_{\boldsymbol{w}}\min_{\boldsymbol{u}}\gamma^2 \tag{2.4}$$

where $\overline{\boldsymbol{S}}_N = \mathrm{diag}([\boldsymbol{S}_N, \boldsymbol{0}, \ldots, \boldsymbol{0}])$.

So far, the problem with time-delays has been transformed into standard discrete H_∞ control problem. Therefore, the former methods and conclusions can be used here. Problems with control-delay can be solved in the same way by introducing the extended vector $\bar{\boldsymbol{x}}_k = \{\boldsymbol{x}_k^{\mathrm{T}}, \boldsymbol{u}_{k-m}^{\mathrm{T}}, \boldsymbol{u}_{k-m+1}^{\mathrm{T}}, \ldots, \boldsymbol{u}_{k-1}^{\mathrm{T}}\}^{\mathrm{T}}$.

3 Controllability of system

Reference [4] presented the concepts of relative and absolute controllability of discrete systems with delays in control, which defines the relative controllability that for any initial state $\boldsymbol{x}_{-m}, \boldsymbol{x}_{-m+1}, \ldots, \boldsymbol{x}_0$ there exists a sequence of controls $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1}$ to satisfy the final state $\boldsymbol{x}_N = \boldsymbol{0}$; while defines the absolute controllability that for any initial extended state $\bar{\boldsymbol{x}}_0$, which is the extended vector introduced above, there exists a sequence of controls $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1}$ to satisfy the final extended state $\bar{\boldsymbol{x}}_N^{\mathrm{T}} = \{\boldsymbol{x}_N^{\mathrm{T}}, \boldsymbol{x}_{N-1}^{\mathrm{T}}, \ldots, \boldsymbol{x}_{N-m}^{\mathrm{T}}\}^{\mathrm{T}} = \boldsymbol{0}$. Obviously the system with time-delays is relatively controllable if it is absolutely controllable. And the conditions of controllability of the extended system are equivalent to the conditions of the absolutely controllability of the original system with time-delays.

4 Computation of optimal induced norm

The problem (2.4) can be expressed as variational form

$$J_c(\boldsymbol{u}, \boldsymbol{w}) = \frac{1}{2}\sum_{k=0}^{N-1}\left(\boldsymbol{z}_k^{\mathrm{T}}\boldsymbol{z}_k - \gamma^2\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{w}_k\right) + \frac{1}{2}\bar{\boldsymbol{x}}_N^{\mathrm{T}}\overline{\boldsymbol{S}}_N\bar{\boldsymbol{x}}_N, \quad \max_{\boldsymbol{w}}\min_{\boldsymbol{u}} J_c, \tag{4.1}$$

which is a min–max problem under constrains. Introducing the Lagrangian multiplier vector $\boldsymbol{\lambda}_{k+1}$ for the system equation (2.2) and directly substituting equation (2.3) into Eq. 4.1, gives the unconstrained variational equation

$$\begin{aligned} J_{cA}(\boldsymbol{u}, \boldsymbol{w}, \bar{\boldsymbol{x}}, \boldsymbol{\lambda}) = \sum_{k=0}^{N-1}\Bigg[&\frac{1}{2}\left(\bar{\boldsymbol{x}}_k^{\mathrm{T}}\overline{\boldsymbol{Q}}\bar{\boldsymbol{x}}_k + \boldsymbol{u}_k^{\mathrm{T}}\boldsymbol{u}_k - \gamma^2\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{w}_k\right) \\ &+ \boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\left(\overline{\boldsymbol{A}}\bar{\boldsymbol{x}}_k + \overline{\boldsymbol{B}}\boldsymbol{u}_k + \overline{\boldsymbol{D}}\boldsymbol{w}_k - \bar{\boldsymbol{x}}_{k+1}\right)\Bigg] + \frac{1}{2}\bar{\boldsymbol{x}}_N^{\mathrm{T}}\overline{\boldsymbol{S}}_N\bar{\boldsymbol{x}}_N. \end{aligned} \tag{4.2}$$

For an arbitrary suboptimal $\gamma^2 > \gamma_{\mathrm{opt}}^2$, the implementation of the variation for vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ gives

$$\boldsymbol{u}_k = -\overline{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1}, \tag{4.3a}$$

$$\boldsymbol{w}_k = \gamma^{-2}\overline{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1} \tag{4.3b}$$

substituting (4.3) back into (4.2), give the variational equation for the dual vectors $\bar{\boldsymbol{x}}$ and $\boldsymbol{\lambda}$

$$J_{cA}(\bar{\boldsymbol{x}},\boldsymbol{\lambda}) = \sum_{k=0}^{N-1}\Bigg[-\Bigg(\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{x}}_{k+1} - \boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\overline{\boldsymbol{A}}\bar{\boldsymbol{x}}_k - \frac{1}{2}\bar{\boldsymbol{x}}_k^{\mathrm{T}}\overline{\boldsymbol{Q}}\bar{\boldsymbol{x}}_k + \frac{1}{2}\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\overline{\boldsymbol{B}}\,\overline{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1}\Bigg) + \frac{1}{2}\gamma^{-2}\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\overline{\boldsymbol{D}}\,\overline{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1}\Bigg] + \frac{1}{2}\bar{\boldsymbol{x}}_N^{\mathrm{T}}\overline{\boldsymbol{S}}_N\bar{\boldsymbol{x}}_N. \tag{4.4}$$

Implement the variational operation and get the dual Hamiltonian difference equations

$$\bar{\boldsymbol{x}}_{k+1} = \overline{\boldsymbol{A}}\bar{\boldsymbol{x}}_k - \left(\overline{\boldsymbol{B}}\,\overline{\boldsymbol{B}}^{\mathrm{T}} - \gamma^{-2}\overline{\boldsymbol{D}}\,\overline{\boldsymbol{D}}^{\mathrm{T}}\right)\boldsymbol{\lambda}_{k+1}, \tag{4.5a}$$

$$\boldsymbol{\lambda}_k = \overline{\boldsymbol{Q}}\bar{\boldsymbol{x}}_k + \overline{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1}. \tag{4.5b}$$

The corresponding boundary conditions are

$$\bar{\boldsymbol{x}}_0 = \boldsymbol{0}, \tag{4.6a}$$

$$\boldsymbol{\lambda}_N = \overline{\boldsymbol{S}}_N\bar{\boldsymbol{x}}_N. \tag{4.6b}$$

Equation.(4.5) can be solved by the routine method of Riccati transformation, that is $\boldsymbol{\lambda}_k = \overline{\boldsymbol{S}}_k\bar{\boldsymbol{x}}_k$

$$\overline{\boldsymbol{S}}_k = \overline{\boldsymbol{Q}} + \overline{\boldsymbol{A}}^{\mathrm{T}}\left(\overline{\boldsymbol{B}}\,\overline{\boldsymbol{B}}^{\mathrm{T}} - \gamma^{-2}\overline{\boldsymbol{D}}\,\overline{\boldsymbol{D}}^{\mathrm{T}} + \overline{\boldsymbol{S}}_{k+1}^{-1}\right)^{-1}\overline{\boldsymbol{A}}, \quad \textit{where } \overline{\boldsymbol{S}}_N \text{ is konwn} \tag{4.7}$$

substituting into (4.3), get the feedback controls as

$$\boldsymbol{u}_k = -\overline{\boldsymbol{B}}^{\mathrm{T}}\overline{\boldsymbol{S}}_{k+1}\bar{\boldsymbol{x}}_{k+1} = -\overline{\boldsymbol{B}}^{\mathrm{T}}\overline{\boldsymbol{S}}_{k+1}\left(\boldsymbol{I} + \left(\overline{\boldsymbol{B}}\,\overline{\boldsymbol{B}}^{\mathrm{T}} - \gamma^{-2}\overline{\boldsymbol{D}}\,\overline{\boldsymbol{D}}^{\mathrm{T}}\right)\overline{\boldsymbol{S}}_{k+1}\right)^{-1}\overline{\boldsymbol{A}}\bar{\boldsymbol{x}}_k, \tag{4.8a}$$

$$\begin{aligned}\boldsymbol{w}_k &= \gamma^{-2}\overline{\boldsymbol{D}}^{\mathrm{T}}\overline{\boldsymbol{S}}_{k+1}\bar{\boldsymbol{x}}_{k+1}\\ &= -\gamma^{-2}\overline{\boldsymbol{D}}^{\mathrm{T}}\overline{\boldsymbol{S}}_{k+1}\left(\boldsymbol{I} + \left(\overline{\boldsymbol{B}}\,\overline{\boldsymbol{B}}^{\mathrm{T}} - \gamma^{-2}\overline{\boldsymbol{D}}\,\overline{\boldsymbol{D}}^{\mathrm{T}}\right)\overline{\boldsymbol{S}}_{k+1}\right)^{-1}\overline{\boldsymbol{A}}\bar{\boldsymbol{x}}_k,\end{aligned} \tag{4.8b}$$

where the induced norm γ^{-2} is involved in Eqs. 4.7 and 4.8, which reflects the demands for robust performance. But Eq. 4.8 has a positive solution only when $\gamma^2 > \gamma_{\mathrm{opt}}^2$, i.e. $\gamma^{-2} < \gamma_{\mathrm{opt}}^{-2}$ is satisfied. Therefore, the solution of $\gamma_{\mathrm{opt}}^{-2}$ is very important.

4.1 Extended Rayleigh-quotient

Variational Equation 4.4 can be expressed as the form

$$\delta\left(\Pi_1 - \gamma^{-2}\Pi_2\right) = 0, \tag{4.9}$$

where

$$\begin{aligned}\Pi_1 = &\sum_{k=0}^{N-1}\left(\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{x}}_{k+1} - \boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\overline{\boldsymbol{A}}\bar{\boldsymbol{x}}_k - \frac{1}{2}\bar{\boldsymbol{x}}_k^{\mathrm{T}}\overline{\boldsymbol{Q}}\bar{\boldsymbol{x}}_k + \frac{1}{2}\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\overline{\boldsymbol{B}\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1}\right)\\ &-\frac{1}{2}\bar{\boldsymbol{x}}_N^{\mathrm{T}}\overline{\boldsymbol{S}}_N\bar{\boldsymbol{x}}_N,\end{aligned} \tag{4.10a}$$

$$\Pi_2 = \frac{1}{2}\sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^{\mathrm{T}} \overline{\boldsymbol{D}}\,\overline{\boldsymbol{D}}^{\mathrm{T}} \boldsymbol{\lambda}_{k+1}, \tag{4.10b}$$

where $\bar{\boldsymbol{x}}$ and $\boldsymbol{\lambda}$ are considered to be two independent variable vectors. Clearly, Π_2 is non-negative while Π_1 should be positive for real solutions. Note that Π_2 is unrelated to $\bar{\boldsymbol{x}}$, so the variational equation

$$\delta_{\boldsymbol{x}}\Pi_1 = 0 \tag{4.11}$$

can be satisfied first, which leads to Eq. 4.5b. Using boundary conditions, one can get

$$\begin{aligned}\Pi_1 &= \sum_{k=0}^{N-1}\left(\bar{\boldsymbol{x}}_k^{\mathrm{T}}\left(\boldsymbol{\lambda}_k - \overline{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1} - \overline{\boldsymbol{Q}}\bar{\boldsymbol{x}}_k\right) + \frac{1}{2}\bar{\boldsymbol{x}}_k^{\mathrm{T}}\overline{\boldsymbol{Q}}\bar{\boldsymbol{x}}_k + \frac{1}{2}\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\overline{\boldsymbol{B}}\,\overline{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1}\right) + \boldsymbol{\lambda}_N^{\mathrm{T}}\bar{\boldsymbol{x}}_N \frac{1}{2}\bar{\boldsymbol{x}}_N^{\mathrm{T}}\overline{\boldsymbol{S}}_f\bar{\boldsymbol{x}}_N,\\ &= \frac{1}{2}\sum_{k=0}^{N-1}\left(\bar{\boldsymbol{x}}_k^{\mathrm{T}}\overline{\boldsymbol{Q}}\bar{\boldsymbol{x}}_k + \boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\overline{\boldsymbol{B}}\,\overline{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{\lambda}_{k+1}\right) + \frac{1}{2}\bar{\boldsymbol{x}}_N^{\mathrm{T}}\overline{\boldsymbol{S}}_f\bar{\boldsymbol{x}}_N > 0 \end{aligned} \tag{4.12}$$

then variational Eq. 4.9 can now be written as

$$\gamma^{-2} = \Pi_1/\Pi_2, \qquad \gamma_{\mathrm{opt}} = \max_{\bar{\boldsymbol{x}}}\min_{\boldsymbol{\lambda}}(\Pi_1/\Pi_2), \tag{4.13}$$

which is considered as extended Rayleigh-quotient, because there are two independent variables $\bar{\boldsymbol{x}}$ and $\boldsymbol{\lambda}$. Based on the analogy between structural mechanics and optimal control theory, γ_{opt}^{-2} corresponds to the fundamental frequency of structural vibration.

If matrix $\overline{\boldsymbol{Q}} = \overline{\boldsymbol{H}}^{\mathrm{T}}\overline{\boldsymbol{H}}$ is positive, variational Eq. 4.13 can be reduced to be the typical Rayleigh-quotient with only one independent variable vector $\boldsymbol{\lambda}$, by substituting $\bar{\boldsymbol{x}}$ using Eq. 4.5b. In the optimal control context, $\overline{\boldsymbol{Q}}$ is not necessarily positive (especially for extended system). But the controllability and observability of system matrices $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \overline{\boldsymbol{H}})$ can ensure the existence of extended Rayleigh-quotient in Eq. 4.13.

For Rayleigh-quotient problems, the extended W–W algorithm enables the eigenvalues to be found to any specified precision [3]. The method is based on the interval mixed energy, which comes from structural mechanics.

4.2 Interval mixed energy

The variational Eq. 4.4 can be expressed as follows

$$J_{cA}(\bar{\boldsymbol{x}}, \boldsymbol{\lambda}) = \sum_{k=0}^{N-1}\left[-\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\bar{\boldsymbol{x}}_{k+1} + V(\bar{\boldsymbol{x}}_k, \boldsymbol{\lambda}_{k+1})\right] + \frac{1}{2}\bar{\boldsymbol{x}}_N^{\mathrm{T}}\overline{\boldsymbol{S}}_f\bar{\boldsymbol{x}}_N, \tag{4.14}$$

where

$$V(\bar{\boldsymbol{x}}_k, \boldsymbol{\lambda}_{k+1}) = \frac{1}{2}\bar{\boldsymbol{x}}_k^{\mathrm{T}}\boldsymbol{E}\bar{\boldsymbol{x}}_k + \boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\overline{\boldsymbol{A}}\bar{\boldsymbol{x}}_k - \frac{1}{2}\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\left(\overline{\boldsymbol{B}}\,\overline{\boldsymbol{B}}^{\mathrm{T}} - \gamma^{-2}\overline{\boldsymbol{D}}\,\overline{\boldsymbol{D}}^{\mathrm{T}}\right)\boldsymbol{\lambda}_{k+1}, \tag{4.15}$$

which defines the interval $(k, k+1)$ mixed energy and has the form for linear problems

$$V(\bar{\boldsymbol{x}}_a, \boldsymbol{\lambda}_b) = \frac{1}{2}\bar{\boldsymbol{x}}_a^{\mathrm{T}}\boldsymbol{E}\bar{\boldsymbol{x}}_a + \boldsymbol{\lambda}_b^{\mathrm{T}}\boldsymbol{F}\bar{\boldsymbol{x}}_a - \frac{1}{2}\boldsymbol{\lambda}_b^{\mathrm{T}}\boldsymbol{G}\boldsymbol{\lambda}_b, \tag{4.16}$$

where $\boldsymbol{E} = \overline{\boldsymbol{Q}}$, $\boldsymbol{F} = \overline{\boldsymbol{A}}$, $\boldsymbol{G} = \overline{\boldsymbol{B}}\,\overline{\boldsymbol{B}}^{\mathrm{T}} - \gamma^{-2}\overline{\boldsymbol{D}}\,\overline{\boldsymbol{D}}^{\mathrm{T}}$ and $\boldsymbol{E}^{\mathrm{T}} = \boldsymbol{E}, \boldsymbol{G}^{\mathrm{T}} = \boldsymbol{G}$.

The importance of interval mixed energy is that two consecutive intervals (t_a, t_b) and (t_b, t_c)can be combined one longer interval (t_a, t_c), whose interval mix energy is $V(t_a, t_c)$

$$V(\bar{\boldsymbol{x}}_a, \boldsymbol{\lambda}_c) = \frac{1}{2}\bar{\boldsymbol{x}}_a^{\mathrm{T}}\boldsymbol{E}_c\bar{\boldsymbol{x}}_a + \boldsymbol{\lambda}_c^{\mathrm{T}}\boldsymbol{F}_c\bar{\boldsymbol{x}}_a - \frac{1}{2}\boldsymbol{\lambda}_c^{\mathrm{T}}\boldsymbol{G}_c\boldsymbol{\lambda}_c, \tag{4.17}$$

where

$$\boldsymbol{E}_c = \boldsymbol{E}_1 + \boldsymbol{F}_1^{\mathrm{T}}(\mathbf{I} + \boldsymbol{E}_2\boldsymbol{G}_1)^{-1}\boldsymbol{E}_2\boldsymbol{F}_1, \tag{4.18a}$$

$$\boldsymbol{G}_c = \boldsymbol{G}_2 + \boldsymbol{F}_2\boldsymbol{G}_1(\mathbf{I} + \boldsymbol{E}_2\boldsymbol{G}_1)^{-1}\boldsymbol{F}_2^{\mathrm{T}}, \tag{4.18b}$$

$$\boldsymbol{F}_c = \boldsymbol{F}_2(\mathbf{I} + \boldsymbol{G}_1\boldsymbol{E}_2)^{-1}\boldsymbol{F}_1. \tag{4.18c}$$

These interval combination equations can be applied recursively (Fig.1). However, in the present context the eigensolutions are mainly concerned. So the eigenvalue count of the interval combination is necessary. The original W–W algorithm is proposed for the case that conditions at both ends are expressed as displacements [5]. While the interval mixed energy is described with $\bar{\boldsymbol{x}}_a$ and $\boldsymbol{\lambda}_b$ at two ends, so the extended W–W algorithm [3] should be used here. For any given $\omega_\# = \gamma_\#^{-2}$, let $J_R(\omega_\#)$ denotes the number of eigenvalue in the interval (t_a, t_b) that smaller than $\omega_\#$ under the end conditions $\bar{\boldsymbol{x}}_a = \mathbf{0}, \boldsymbol{\lambda}_b = \mathbf{0}$. From the method described in [3], the eigenvalue count for interval combination is given

$$J_{Rc}(\omega_\#) = J_{R1}(\omega_\#) + J_{R2}(\omega_\#) - s\{\boldsymbol{E}_2\} + s\{\boldsymbol{G}_1 + \boldsymbol{E}_2^{-1}\}, \tag{4.19}$$

where $s\{\boldsymbol{M}\}$denotes the number of negative elements in matrix $\boldsymbol{D}$, where matrix $\boldsymbol{M}$ is factorized as $\boldsymbol{M} = \boldsymbol{LDL}^{\mathrm{T}}$

Therefore, the interval mixed energy used in the eigenvalue problem should be expressed as $(\boldsymbol{E}, \boldsymbol{G}, \boldsymbol{F}, J_R(\omega_\#))$, all of which are functions of $(t_a, t_b)$and $\omega_\# = \gamma_\#^{-2}$. And initial conditions are $\boldsymbol{E} \to \mathbf{O}, \boldsymbol{G} \to \mathbf{O}, \boldsymbol{F} \to \mathbf{I}$, when $t_a \to t_b$

Let $(\boldsymbol{E}_1, \boldsymbol{G}_1, \boldsymbol{F}_1, J_{R1}(\omega_\#))$denotes the interval$\left(t_k, t_{k+1}\right)$, and $(\boldsymbol{E}_2, \boldsymbol{G}_2, \boldsymbol{F}_2, J_{R2}(\omega_\#))$ denotes the interval $\left(t_{k+1}, t_N\right)$ and $(\boldsymbol{E}_{\mathrm{c}}, \boldsymbol{G}_{\mathrm{c}}, \boldsymbol{F}_{\mathrm{c}}, J_{Rc}(\omega_\#))$denotes the interval (t_k, t_N), then the combination equation (4.18a) is the same as the discrete Riccati iterative equation (4.7) except for the initial conditions. Actually implement the following equation after interval combination (t_k, t_N)

$$\overline{\boldsymbol{S}}_k = \boldsymbol{E}_c + \boldsymbol{F}_c^{\mathrm{T}}(\overline{\boldsymbol{S}}_N^{-1} + \boldsymbol{G}_c)^{-1}\boldsymbol{F}_c, \tag{4.20}$$

$$J_{RNc}(\omega_\#) = J_c(\omega_\#) - s\{\overline{\boldsymbol{S}}_f\} + s\{\overline{\boldsymbol{S}}_f^{-1} + \boldsymbol{G}_c\}, \tag{4.21}$$

where $\overline{\boldsymbol{S}}_k$ is the solution of Eq. 4.7 at t_k. If $J_{RNc}(\omega_\#) = 0$, the specified $\gamma_\#^{-2} = \omega_\#$ is a suboptimal parameter, otherwise $\gamma_\#^{-2}$ is too large hence no positive solution of Riccati equation (4.7) exists. Based on the criterion, some searching procedures, such as bisection method, can be introduced to find the lowest eigenvalue $\gamma_{\mathrm{opt}}^{-2}$ to any required precision.

4.3 Procedure of optimal induced norm

The optimal induced norm is the result of a bisection method of suboptimal solutions. The algorithm is given below in meta-language.

Firstly compute the extended matrices $\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \overline{\boldsymbol{D}}, \overline{\boldsymbol{H}}, \overline{\boldsymbol{N}}$ from the original delay-system matrices $\boldsymbol{A}_0, \boldsymbol{B}_0, \boldsymbol{D}_0, \boldsymbol{H}_0, \boldsymbol{N}_0$, then enter into the following recursive procedure to search γ_{opt}^{-2}:

0. { Select a suitable $\gamma_{\#}^{-2}$; compute $\boldsymbol{F} = \overline{\boldsymbol{A}}$, $\boldsymbol{G} = \overline{\boldsymbol{B}\boldsymbol{B}}^{\mathrm{T}} - \gamma^{-2}\overline{\boldsymbol{D}\boldsymbol{D}}^{\mathrm{T}}, \boldsymbol{E} = \overline{\boldsymbol{H}}^{\mathrm{T}}\overline{\boldsymbol{H}}$ }.
1. { $\boldsymbol{E}_1 = \boldsymbol{E}_2 = \boldsymbol{E}$; $\boldsymbol{G}_1 = \boldsymbol{G}_2 = \boldsymbol{G}$; $\boldsymbol{F}_1 = \boldsymbol{F}_2 = \boldsymbol{F}$; $J_{R1} = J_{R2} = 0$ }.
2. For (k=1;$k \le N-1$;k++) { ***comment***: use $k \in [0, N-1]$ for finite horizon case; while use $\|\boldsymbol{F}_c\| \to 0$ for infinite horizon case;
   {Compute $\boldsymbol{E}_c$, $\boldsymbol{G}_c$, $\boldsymbol{F}_c$ and J_{Rc} from Eqs. 4.18a–c and 4.19 }
   {$\boldsymbol{E}_2 = \boldsymbol{E}_c$; $\boldsymbol{G}_2 = \boldsymbol{G}_c$; $\boldsymbol{F}_2 = \boldsymbol{F}_c$; $J_{R2} = J_{Rc}$}
   }
   {Compute J_{RNc} from Eq. 4.21 } ***comment***: γ_{opt}^{-2} for infinite horizon case is independent to boundary conditions, the segment can be omitted.
3. If (J_{RNc} >0)
   {$\gamma_{\#}^{-2}$ is an upper bound (**ub**), and should be lower in the next iteration }
   else
   {$\gamma_{\#}^{-2}$ is a lower bound (**lb**), and should be upper in the next iteration }
   if (**ub**-**lb**) $> \varepsilon$ (ε is the specified precision)
   {reatart from step 0 with the modified $\gamma_{\#}^{-2}$}
   else
   { break }

The iteration for $\gamma_{\#}^{-2}$ should be continued until the specified precision is reached. The lower bound (**lb**) is taken as γ_{opt}^{-2}.

5 Examples

Example 1:

This example is taken from Ref. [6], which is a continuous system with state delays. The sampling period is $T = 0.125$, using zero-order holder, get the corresponding discrete data as follows

$$\boldsymbol{A}_0 = \begin{bmatrix} 0.687289 & 0.1114648 \\ 0 & 1.1331485 \end{bmatrix}, \quad \boldsymbol{A}_m = \lambda \cdot \begin{bmatrix} 0.1049597 & 0.1042369 \\ 0.01331485 & 0 \end{bmatrix},$$
$$\boldsymbol{B}_0 = \begin{bmatrix} 0.118693 \\ 0.2662969 \end{bmatrix}, \qquad \boldsymbol{D}_0 = \begin{bmatrix} 0.1042369 \\ 0 \end{bmatrix}, \qquad \boldsymbol{H}_0^{\mathrm{T}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

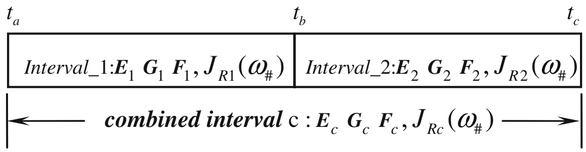
where λ denotes the weight of the matrix of delayed state, m denotes the number of delay period.

Figures 2 and 3 give the curves of optimal induced norm versus to λ and m (state-delay), respectively

Example 2:

This example is taken from Ref. [6], which is a continuous system with control delays. The sampling period is $T = 0.125$, using zero-order holder, get the corresponding discrete data as follows

Fig. 1 The interval combination of mixed energy



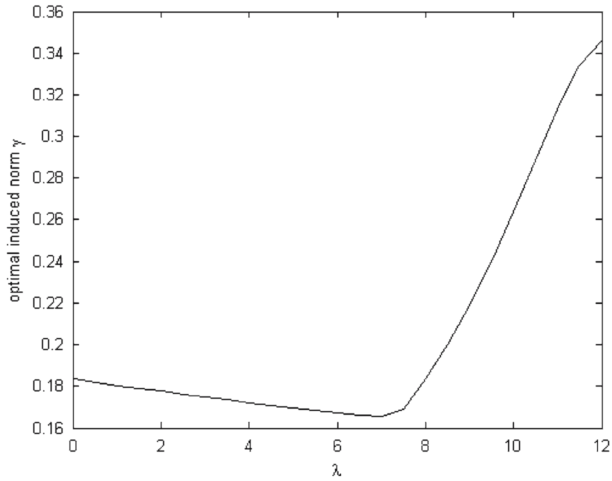


Fig. 2 With state delay ($m = 2$)

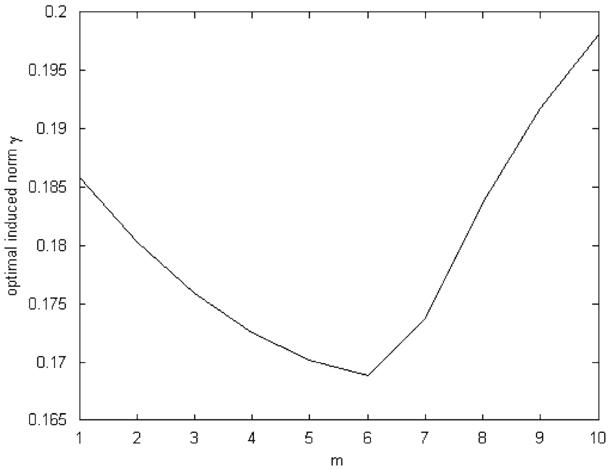


Fig. 3 With state delay ($\lambda = 1.0$)

$$\boldsymbol{A}_0 = \begin{bmatrix} 0.778801 & 0.118116 \\ 0 & 1.1331485 \end{bmatrix}, \qquad \boldsymbol{B}_0 = \begin{bmatrix} 0.00751628 \\ 0.1331485 \end{bmatrix},$$

$$\boldsymbol{B}_m = \lambda \cdot \begin{bmatrix} 0.02211992 \\ 0 \end{bmatrix}, \quad \boldsymbol{D}_0 = \begin{bmatrix} 0.1105996 \\ 0 \end{bmatrix}, \qquad \boldsymbol{H}_0^{\mathrm{T}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where λ denotes the weight of the matrix of delayed control, m denotes the number of delay period.

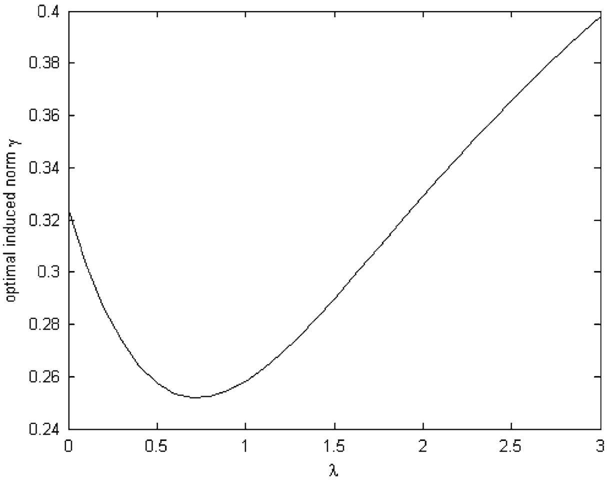


Fig. 4 With control delay ($m = 2$)

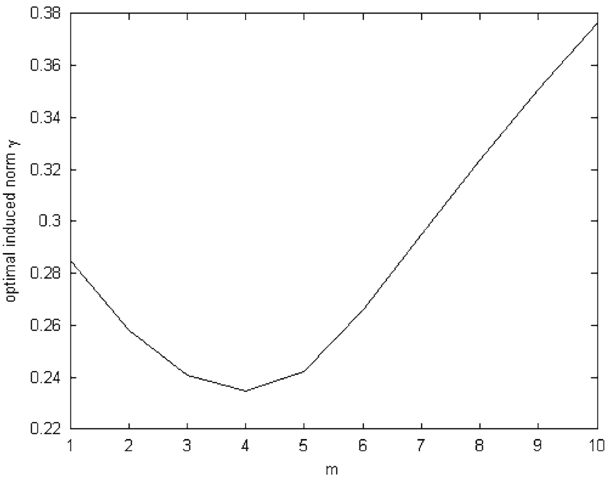


Fig. 5 With control delay ($\lambda = 1.0$)

Figures 4 and 5 give the curves of optimal induced norm versus to λ and m (control-delay), respectively.

Numerical results demonstrate that the optimal norm does not increase monotonously with time-delays and can be decreased effectively by selecting appropriate time-delays.

6 Concluding remarks

The optimal H_∞ induced norm computation for discrete system with time-delays is investigated in this paper. Introducing the extended vector, the original equations with time-delays are transformed into standard equations. Then the former conclusions can be used. Based on analogies between structural mechanics and optimal control theory, the optimal norm corresponds to the fundamental frequency of structural vibration, which is a Rayleigh-quotient problem and can be solved by extended W–W algorithm. The controller is deduced from the original equations with time-delays, without any approximation, so system stability can be guaranteed and the optimal norm is dependent on time delays. Numerical results demonstrate that the optimal norm does not increase monotonously with time-delays and can be decreased effectively by selecting appropriate time-delays. And the method can be extended to the H_∞filtering problem with time-delays conveniently.

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